CHAPTER

21 Mathematical Reasoning

Mathematical Statements

A statement is a sentence which is either true or false, but not both simultaneously.

Example

1. "I'm happy today !" is not a statement
2. "New delhi is in India", it is a statement
3. "x is a natural number", is not a statement

Simple Statement

A statement is simple if it can not be broken down into two or more statement.

Compound Statement

A compound statement is a statement which is made up of two or more statements. The individual statements are called component statements.

E.g., "All rational numbers are real and all real numbers are complex". Is a compound statement.

The two component statements are:

p : All rational numbers are real.

q : All real numbers are complex.

Negation of a Statement

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$ and read as 'not p'.

E.g., Consider a statement:

p : sky is blue

Its negation is:

$\sim p$: sky is not blue"

Connectives

The word 'and' (conjunction) ($\wedge$)

Any two component statements can be connected by the word 'and' to form a compound statement.

The rules for the compound statement with 'and' are:

Rule 1: The compound statement with 'and' is true if all its component statements are true.

Rule 2: The compound statement with 'and' is false if some or all of its component statements are false.

The word 'or' (disjunction) ($\vee$)

Any two component statements can be connected by the word "or" to form a compound statement.

The rules for the compound statement with 'or' are:

Rule 1: A compound statement with an 'or' is true if one component statement is true or all the component statements are true.

Rule 2: A compound statement with an 'or' is false if all the component statements are false.

Quantifiers

In mathematics, sometimes we come across many mathematical statements containing phrases **"There exists"** and **"For every"**. These two phrases are called **quantifiers**. Phrase **"for every (or for all)"** is called the **universal quantifier** and the phrase **"There exists"** is known as the **existential quantifier**.

E.g., p : For every real number x, x is less than $x + 1$.

q : There exists a capital for every country in the world.

Implications

The statements of the form "If p then q", "p only if q", and "if and only if" are called implications.

p implies q is denoted by $p \Rightarrow q \equiv \sim p \vee q$

if and only if is denoted by the symbol $\Leftrightarrow$

General Rules for Validating Statements

Rule 1: If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.

Case I: Show that the statement p is true.

Case II: Show that the statement q is true.

Rule 2: In order to show that the statement "p or q" is true, one must consider the following.

Case I: By assuming that p is false, show that q must be true.

Case II: By assuming that q is false, show that p must be true.

Rule 3: Statements with 'If-then'

In order to prove the statement "If p then q" we need to show that any of the following cases is true.

Case I: By assuming that p is true, show that q must be true.

Case II: By assuming that q is false, show that p must be false.

Rule 4: Statements with 'If and only if'

In order to prove the statement "p if and only if q", we need to show.

(i) If p is true, then q is true and

(ii) If q is true, then p is true.

Method of Contradiction

To check whether a statement p is true, we assume that p is not true *i.e.* $\sim p$ is true. Then, we arrive at some result which is contradictory to our assumption. Therefore, we conclude that p is true.

This method involves giving an example of a situation where the statement is not valid. Such an example is called a counter example.

Important /Critical Points to Remember

Truth Table

A table that shows the relationship between the truth value of compound statement. $S(p, q, r, ...)$ and the truth values of its sub-statements $p, q, r, ...$, etc., is called the truth table of statement S. If p and q are two simple statements then truth table for basic logical connectives of:

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$\sim p$
T	F
F	T

Conditional

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$ or $(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Two compound statements $S_1(p_1, q_1, r_1, ...)$ and $S_2(p_2, q_2, r_2, ...)$ are said to be logically equivalent if they have the same truth values for all logical possibilities that is identical truth table.

If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

Tautology: A statement is said to be a tautology if it is true for all logical possibilities.

Contradiction: A statement is a contradiction if it is false for all logical possibilities *i.e.* its truth value is always F.

Duality

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ *by* $\vee$ and $\vee$ *by* $\wedge$.

Converse: The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$.

Inverse: The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$

Contrapositive: The contrapositive of the conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

Negation of Compound Statements

(*a*) **Negation of conjunction:** If p and q are two statements then

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

(*b*) **Negation of disconjunction:** If p and q are two statements then $\sim (p \vee q) \equiv \sim p \wedge \sim q$

(*c*) **Negation of implication:**

If p and q are two statements, then $\sim (p \Rightarrow q) \equiv p \wedge \sim q$

(*d*) **Negation of Biconditional:**

If p and q are two statements, then

$\sim (p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Algebra of Statements

If p, q, r are any three statements then some law of algebra of statements are as follow:

(*a*) **Idempotent Laws**

(*i*) $p \vee p \equiv p$

(*ii*) $p \wedge p \equiv p$

(*b*) **Commutative Laws**

(*i*) $p \vee q \equiv q \vee p$

(*ii*) $p \wedge q \equiv q \wedge p$

(*c*) **Associative Law**

(*i*) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(*ii*) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(*d*) **Distributive Laws**

(*i*) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(*ii*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(*e*) **De'Morgan's Law**

(*i*) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(*ii*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

(*f*) **Contrapositive Laws**

For any statement p, we have

$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(*g*) **Involution Laws (Double Negation Laws)**

$\sim(\sim p) \equiv p$